GEOMETRIC REALIZATIONS OF PARA-HERMITIAN CURVATURE MODELS

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Abstract. We show that a para-Hermitian algebraic curvature model satisfies the para-Gray identity if and only if it is geometrically realizable by a para-Hermitian manifold. This requires extending the Tricerri-Vanhecke curvature decomposition to the para-Hermitian setting. Additionally, the geometric realization can be chosen to have constant scalar curvature and constant $\star$-scalar curvature.

This paper is dedicated to the memory of **Professor Katsumi Nomizu**

1. Introduction

1.1. **Hermitian geometry.** Let g be a Riemannian metric on a smooth manifold M of dimension $2n$. Let $\mathcal{J}$ give (M,g) an *almost Hermitian* structure. This means that $\mathcal{J}$ is an almost complex structure on the tangent bundle which is compatible with g, i.e. $\mathcal{J}^2=-\operatorname{id}$ and $\mathcal{J}^*g=g$. We say that the *almost Hermitian* manifold

$$\mathcal{M}:=(M,g,\mathcal{J})$$

is *Hermitian* if $\mathcal{J}$ is integrable, i.e. if the Nijenhuis tensor vanishes or, equivalently, there exist local coordinates $(x_1,...,x_n,y_1,...,y_n)$ centered at any given point of the manifold so that

$$\mathcal{J}\partial_{x_i}=\partial_{y_i}\quad\text{and}\quad\mathcal{J}\partial_{y_i}=-\partial_{x_i}\,.$$

We refer to [5] for further details.

The Riemann curvature tensor

$$R(x,y):=\nabla_x\nabla_y-\nabla_y\nabla_x-\nabla_{[x,y]}$$

of the Levi-Civita connection [6] satisfies:

$$\begin{aligned}&R(x,y,z,w)+R(y,z,x,w)+R(z,x,y,w)=0,\\&R(x,y,z,w)=-R(y,x,z,w)=R(z,w,x,y)\,.\end{aligned}\tag{1.a}$$

Gray [4] showed that there is an additional identity, which is called the Gray identity, which is satisfied by the curvature tensor of any Hermitian manifold:

$$\begin{aligned}0\;=\;&R(x,y,z,w)+R(Jx,Jy,Jz,Jw)-R(Jx,Jy,z,w)\\-\;&R(Jx,y,Jz,w)-R(Jx,y,z,Jw)-R(x,Jy,Jz,w)\\-\;&R(x,Jy,z,Jw)-R(x,y,Jz,Jw)\,.\end{aligned}\tag{1.b}$$

All universal curvature symmetries for Hermitian manifolds are generated by the relations of Equations (1.a) and (1.b). By contrast, there are no additional symmetries beyond those of Equation (1.a) in the almost Hermitian context. One can make this statement precise as follows. Let $\langle\cdot,\cdot\rangle$ be a positive definite inner product on a real vector space V of dimension $2n$. Let J be a Hermitian complex

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structure on V; $J^2 = -\mathrm{id}$ and $J^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$. Let $A \in \otimes^4 V^*$ be an *algebraic curvature tensor*, i.e. A satisfies the symmetries of Equation (1.a). Let

$$\mathfrak{C} := (V, \langle\cdot,\cdot\rangle, J, A)$$

be the associated *Hermitian curvature model.* We say that $\mathfrak{C}$ is geometrically realized by an almost Hermitian manifold $\mathcal{M} = (M, g, \mathcal{J})$ if there is an isomorphism $\phi : V \to T_P M$ for some $P \in M$ so that $\phi^* g_P = \langle\cdot,\cdot\rangle$, $\phi^*\mathcal{J}_P = J$, and $\phi^* R_P = A$. We refer to [1, 2] for the proof of the following result:

Theorem 1.1. *Let $\mathfrak{C}$ be a Hermitian curvature model.*

(1) *$\mathfrak{C}$ is always geometrically realized by an almost Hermitian manifold.*
(2) *$\mathfrak{C}$ is geometrically realized by a Hermitian manifold if and only if $\mathfrak{C}$ satisfies* Equation (1.b).

There are analogous questions in the affine setting. For example, if ∇ is both holomorphic and affine Kaehler, then $R = 0$ and ∇ is locally flat [7].

1.2. **Para-Hermitian geometry.** Let $(\tilde{M}, \tilde{g})$ be a pseudo-Riemannian manifold of dimension $2n$. Let $\tilde{\mathcal{J}}$ give $(\tilde{M}, \tilde{g})$ an *almost para-Hermitian* structure; $\tilde{\mathcal{J}}^2 = \mathrm{id}$ and $\tilde{\mathcal{J}}^*\tilde{g} = -\tilde{g}$. In this setting, necessarily $\tilde{g}$ has neutral signature (n, n). The *almost para-Hermitian* manifold

$$\tilde{\mathcal{M}} := (\tilde{M}, \tilde{g}, \tilde{\mathcal{J}})$$

is said to be *para-Hermitian* if $\tilde{\mathcal{J}}$ is integrable, i.e. if the Nijenhuis tensor $N_{\tilde{\mathcal{J}}}$ vanishes (see, for instance, [3]), where

$$N_{\tilde{\mathcal{J}}}(x, y) := [x, y] - \tilde{\mathcal{J}}[\tilde{\mathcal{J}}x, y] - \tilde{\mathcal{J}}[x, \tilde{\mathcal{J}}y] + [\tilde{\mathcal{J}}x, \tilde{\mathcal{J}}y].$$

Equivalently, there exist local coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ centered at any given point of $\tilde{M}$ so that

$$\tilde{\mathcal{J}}\partial_{x_i} = \partial_{y_i} \quad \text{and} \quad \tilde{\mathcal{J}}\partial_{y_i} = \partial_{x_i}\,.$$

In the algebraic setting, let $\widetilde{\langle\cdot,\cdot\rangle}$ be a neutral signature inner product on a finite dimensional vector space $\tilde{V}$. Let $\tilde{J}$ be a *para-Hermitian structure* on $(\tilde{V}, \widetilde{\langle\cdot,\cdot\rangle})$, i.e. $\tilde{J}^2 = \mathrm{id}$ and $\tilde{J}^*\widetilde{\langle\cdot,\cdot\rangle} = -\widetilde{\langle\cdot,\cdot\rangle}$. If $\tilde{A} \in \otimes^4 \tilde{V}^*$ is an algebraic curvature tensor, let

$$\tilde{\mathfrak{C}} := (\tilde{V}, \widetilde{\langle\cdot,\cdot\rangle}, \tilde{J}, \tilde{A})$$

be the corresponding *para-Hermitian curvature model.* We change the signs in Equation (1.b) to define a corresponding *para-Gray* relation

$$\begin{aligned} 0 \;&=\; \tilde{A}(x, y, z, w) + \tilde{A}(\tilde{J}x, \tilde{J}y, \tilde{J}z, \tilde{J}w) + \tilde{A}(\tilde{J}x, \tilde{J}y, z, w) \\ &+\; \tilde{A}(\tilde{J}x, y, \tilde{J}z, w) + \tilde{A}(\tilde{J}x, y, z, \tilde{J}w) + \tilde{A}(x, \tilde{J}y, \tilde{J}z, w) \\ &+\; \tilde{A}(x, \tilde{J}y, z, \tilde{J}w) + \tilde{A}(x, y, \tilde{J}z, \tilde{J}w)\,. \end{aligned} \tag{1.c}$$

Assertion (1) in the following Theorem was established in [1]; Assertion (2) is the main new result of this paper:

Theorem 1.2. *Let $\tilde{\mathfrak{C}}$ be a para-Hermitian curvature model.*

(1) *$\tilde{\mathfrak{C}}$ is always geometrically realized by an almost para-Hermitian manifold.*
(2) *$\tilde{\mathfrak{C}}$ is geometrically realized by a para-Hermitian manifold if and only if $\tilde{\mathfrak{C}}$ satisfies* Equation (1.c).

Remark 1.3. We make the following observations:

(1) The results of [1] show that the manifolds in Theorems 1.1 and 1.2 can be chosen to have constant scalar curvature and constant $\star$-scalar curvature.
(2) The methods we will develop to establish Theorem 1.2 (2) can be used to show that Theorem 1.1 holds for pseudo-Riemannian manifolds; it is not necessary to assume that the inner product is positive definite.

(3) In the Hermitian setting, let $\Omega(\cdot,\cdot) := g(\cdot,\mathcal{J}\cdot)$ be the Kaehler form; in the para-Hermitian setting, the para-Kaehler form is defined similarly by setting $\tilde{\Omega}(\cdot,\cdot) := \tilde{g}(\cdot,\tilde{\mathcal{J}}\cdot)$. The geometric realizations can be chosen so that $d\Omega_P = 0$ in the Hermitian setting or $d\tilde{\Omega}_P = 0$ in the para-Hermitian setting. Thus requiring the Kaehler or the para-Kaehler identity (i.e. $d\Omega = 0$ or $d\tilde{\Omega} = 0$) at a single point imposes no additional curvature restrictions although, of course requiring the Kaehler identity globally yields additional curvature restrictions.

1.3. **Outline of the paper.** Here is a brief outline to the paper. In Section 2, we will show that the curvature tensor of any para-Hermitian manifold satisfies Equation (1.c) and thereby establish one implication of Theorem 1.2 (2). Rather than generalizing Gray's proof from the Hermitian to the para-Hermitian setting, we have chosen to give a direct proof which is quite different in flavor. In Section 3, we recall the Tricerri-Vanhecke [8] decomposition of the space of algebraic curvature tensors in the Hermitian setting and extend it to the para-Hermitian setting by complexification; this result is perhaps of interest in its own right. In Section 4, we linearize the problem. We define a linear subspace $\mathfrak{P}$ of the space of all algebraic curvature tensors which is invariant under the para-unitary structure group such that any element of $\mathfrak{P}$ can be realized by a para-Hermitian metric with vanishing Kaehler form at the point in question. We complete the proof of Theorem 1.2 (2) in Section 5 by showing the elements of $\mathfrak{P}$ are precisely those algebraic curvature tensors which satisfy the para-Gray identity given in Equation (1.c).

2. The para-Gray identity for para-Hermitian manifolds

Let $\tilde{J}$ be a para-Hermitian structure on $(\tilde{V},\widetilde{\langle\cdot,\cdot\rangle})$. Let $\{\tilde{e}_a\}$ be a basis for $\tilde{V}$. If $\tilde{T}\in\otimes^4\tilde{V}^*$, we define the *para-Gray symmetrization*

$$\begin{aligned}\tilde{\mathcal{G}}(\tilde{T})(\tilde{e}_a,\tilde{e}_b,\tilde{e}_c,\tilde{e}_d) :&= \tilde{T}(\tilde{e}_a,\tilde{e}_b,\tilde{e}_c,\tilde{e}_d)+\tilde{T}(\tilde{J}\tilde{e}_a,\tilde{J}\tilde{e}_b,\tilde{J}\tilde{e}_c,\tilde{J}\tilde{e}_d)\\&+\tilde{T}(\tilde{J}\tilde{e}_a,\tilde{J}\tilde{e}_b,\tilde{e}_c,\tilde{e}_d)+\tilde{T}(\tilde{J}\tilde{e}_a,\tilde{e}_b,\tilde{J}\tilde{e}_c,\tilde{e}_d)\\&+\tilde{T}(\tilde{J}\tilde{e}_a,\tilde{e}_b,\tilde{e}_c,\tilde{J}\tilde{e}_d)+\tilde{T}(\tilde{e}_a,\tilde{J}\tilde{e}_b,\tilde{J}\tilde{e}_c,\tilde{e}_d)\\&+\tilde{T}(\tilde{e}_a,\tilde{J}\tilde{e}_b,\tilde{e}_c,\tilde{J}\tilde{e}_d)+\tilde{T}(\tilde{e}_a,\tilde{e}_b,\tilde{J}\tilde{e}_c,\tilde{J}\tilde{e}_d)\,.\end{aligned}$$

We establish one implication of Theorem 1.2 (2) by showing:

Theorem 2.1. *If $\tilde{\mathcal{M}}=(\tilde{M},\tilde{g},\tilde{\mathcal{J}})$ is a para-Hermitian manifold, then $\tilde{\mathcal{G}}(\tilde{R})=0$.*

Proof. Introduce coordinates $(u_1,...,u_{2n})$ on $\tilde{M}$ so

$$\tilde{\mathcal{J}}\partial_{u_1}=\partial_{u_{n+1}},\ \dots\ ,\tilde{\mathcal{J}}\partial_{u_n}=\partial_{u_{2n}},\ \tilde{\mathcal{J}}\partial_{u_{n+1}}=\partial_{u_1},\ \dots\ ,\tilde{\mathcal{J}}\partial_{u_{2n}}=\partial_{u_n}\,.$$

We shall let indices $a,b,c,\dots$ range from 1 to $2n$ and index the coordinate frame $\{\xi_1,...,\xi_{2n}\}:=\{\partial_{u_1},...,\partial_{u_{2n}}\}$. We also let indices $\alpha,\beta,\gamma,\dots$ range from 1 to $2n$. Let

$$\tilde{g}_{ab}:=\tilde{g}(\xi_a,\xi_b),\quad \tilde{g}_{\alpha\beta}:=\tilde{g}(\tilde{J}\xi_a,\tilde{J}\xi_b),\quad \tilde{g}_{a\beta}:=\tilde{g}(\xi_a,\tilde{J}\xi_b),\quad \tilde{g}_{\alpha b}:=\tilde{g}(\tilde{J}\xi_a,\xi_b)\,.$$

We have $\tilde{g}_{ab}=-\tilde{g}_{\alpha\beta}$ and $\tilde{g}_{a\beta}=-\tilde{g}_{\alpha b}$. Let $\tilde{g}^{ab}$ be the inverse matrix. We adopt the *Einstein* convention and sum over repeated indices. Let "/" denote ordinary partial differentiation. Let $\tilde{\Gamma}$ be the Christoffel symbols of the Levi-Civita connection. We compute:

$$\begin{aligned}&\tilde{\Gamma}_{abc}=\tfrac12(\tilde{g}_{bc/a}+\tilde{g}_{ac/b}-\tilde{g}_{ab/c}),\qquad \tilde{\Gamma}_{ab}{}^d=\tilde{g}^{cd}\tilde{\Gamma}_{abc},\\&\tilde{R}_{abc}{}^d=\partial_{u_a}\tilde{\Gamma}_{bc}{}^d-\partial_{u_b}\tilde{\Gamma}_{ac}{}^d+\tilde{\Gamma}_{ae}{}^d\tilde{\Gamma}_{bc}{}^e-\tilde{\Gamma}_{be}{}^d\tilde{\Gamma}_{ac}{}^e\,.\end{aligned}$$

This enables us to compute:

$$\begin{aligned}\tilde R_{abcd} &= \tilde g_{df}\partial_{u_a}\tilde\Gamma_{bc}{}^f - \tilde g_{df}\partial_{u_b}\tilde\Gamma_{ac}{}^f + \tilde g_{df}\tilde\Gamma_{ae}{}^f\tilde\Gamma_{bc}{}^e - \tilde g_{df}\tilde\Gamma_{be}{}^f\tilde\Gamma_{ac}{}^e\\ &= \tilde\Gamma_{bcd/a} - \tilde g_{df/a}\tilde\Gamma_{bc}{}^f - \tilde\Gamma_{acd/b} + \tilde g_{df/b}\tilde\Gamma_{ac}{}^f + \tilde g^{el}\tilde\Gamma_{aed}\tilde\Gamma_{bcl} - \tilde g^{el}\tilde\Gamma_{bed}\tilde\Gamma_{acl}\\ &= \tilde\Gamma_{bcd/a} - \tilde g^{fl}\tilde g_{df/a}\tilde\Gamma_{bcl} - \tilde\Gamma_{acd/b} + \tilde g^{fl}\tilde g_{df/b}\tilde\Gamma_{acl} + \tilde g^{el}\tilde\Gamma_{aed}\tilde\Gamma_{bcl} - \tilde g^{el}\tilde\Gamma_{bed}\tilde\Gamma_{acl}.\end{aligned}$$

We first study the linear terms in the second derivatives of the metric:

$$\tilde\Gamma_{bcd/a} - \tilde\Gamma_{acd/b} = \tfrac12\{\tilde g_{bd/ac} + \tilde g_{ac/bd} - \tilde g_{bc/ad} - \tilde g_{ad/bc}\}\,.$$

We examine the role $\tilde T^1_{abcd} := \tilde g_{bd/ac}$ plays in the para-Gray identity; the remaining 3 terms play similar roles and the argument is similar after permuting the indices appropriately. We use the fact that $\tilde J^*\tilde g = -\tilde g$ and apply $\tilde{\mathcal G}$ to compute

$$\begin{aligned}\tilde{\mathcal G}(\tilde T^1)_{abcd} &= \tilde g_{bd/ac} + \tilde g_{\beta\delta/\alpha\gamma} + \tilde g_{\beta d/\alpha c} + \tilde g_{bd/\alpha\gamma}\\ &+ \tilde g_{b\delta/\alpha c} + \tilde g_{\beta d/a\gamma} + \tilde g_{\beta\delta/ac} + \tilde g_{b\delta/a\gamma}\\ &= \tilde g_{bd/ac} - \tilde g_{bd/\alpha\gamma} - \tilde g_{b\delta/\alpha c} + \tilde g_{bd/\alpha\gamma}\\ &+ \tilde g_{b\delta/\alpha c} - \tilde g_{b\delta/a\gamma} - \tilde g_{bd/ac} + \tilde g_{b\delta/a\gamma}\\ &= 0\,.\end{aligned}$$

Next we examine the terms which are quadratic in the first derivatives of the metric; there are three different kinds of terms which must be symmetrized:

$$\tilde T^2_{abcd} := \tilde g^{fe}\tilde g_{ad/f}\tilde g_{bc/e},\quad \tilde T^3_{abcd} := \tilde g^{fe}\tilde g_{af/d}\tilde g_{bc/e},\quad \tilde T^4_{abcd} := \tilde g^{fe}\tilde g_{af/d}\tilde g_{be/c}\,.$$

The remaining quadratic terms arise by permuting the roles of $\{a,b,c,d\}$ in these expressions. We compute:

$$\begin{aligned}\tilde{\mathcal G}(\tilde T^2)_{abcd} &= \tilde g^{fe}\{\tilde g_{ad/f}\tilde g_{bc/e} + \tilde g_{\alpha\delta/f}\tilde g_{\beta\gamma/e} + \tilde g_{\alpha d/f}\tilde g_{\beta c/e} + \tilde g_{\alpha d/f}\tilde g_{b\gamma/e}\\ &\qquad + \tilde g_{\alpha\delta/f}\tilde g_{bc/e} + \tilde g_{ad/f}\tilde g_{\beta\gamma/e} + \tilde g_{a\delta/f}\tilde g_{\beta c/e} + \tilde g_{a\delta/f}\tilde g_{b\gamma/e}\}\\ &= \tilde g^{fe}\{\tilde g_{ad/f}\tilde g_{bc/e} + \tilde g_{ad/f}\tilde g_{bc/e} + \tilde g_{a\delta/f}\tilde g_{b\gamma/e} - \tilde g_{a\delta/f}\tilde g_{b\gamma/e}\\ &\qquad - \tilde g_{ad/f}\tilde g_{bc/e} - \tilde g_{ad/f}\tilde g_{bc/e} - \tilde g_{a\delta/f}\tilde g_{b\gamma/e} + \tilde g_{a\delta/f}\tilde g_{b\gamma/e}\} = 0,\\ \tilde{\mathcal G}(\tilde T^3)_{abcd} &= \tilde g^{fe}\{\tilde g_{af/d}\tilde g_{bc/e} + \tilde g_{\alpha f/\delta}\tilde g_{\beta\gamma/e} + \tilde g_{\alpha f/d}\tilde g_{\beta c/e} + \tilde g_{\alpha f/d}\tilde g_{b\gamma/e}\\ &\qquad + \tilde g_{\alpha f/\delta}\tilde g_{bc/e} + \tilde g_{af/d}\tilde g_{\beta\gamma/e} + \tilde g_{af/\delta}\tilde g_{\beta c/e} + \tilde g_{af/\delta}\tilde g_{b\gamma/e}\}\\ &= \tilde g^{fe}\{\tilde g_{af/d}\tilde g_{bc/e} - \tilde g_{\alpha f/\delta}\tilde g_{bc/e} - \tilde g_{\alpha f/d}\tilde g_{b\gamma/e} + \tilde g_{\alpha f/d}\tilde g_{b\gamma/e}\\ &\qquad + \tilde g_{\alpha f/\delta}\tilde g_{bc/e} - \tilde g_{af/d}\tilde g_{bc/e} - \tilde g_{af/\delta}\tilde g_{b\gamma/e} + \tilde g_{af/\delta}\tilde g_{b\gamma/e}\} = 0.\end{aligned}$$

The final term requires a bit more work.

$$\begin{aligned}\tilde{\mathcal G}(\tilde T^4)_{abcd} &= \tilde g^{fe}\{\tilde g_{af/d}\tilde g_{be/c} + \tilde g_{\alpha f/\delta}\tilde g_{\beta e/\gamma} + \tilde g_{\alpha f/d}\tilde g_{\beta e/c} + \tilde g_{\alpha f/d}\tilde g_{be/\gamma}\\ &\qquad + \tilde g_{\alpha f/\delta}\tilde g_{be/c} + \tilde g_{af/d}\tilde g_{\beta e/\gamma} + \tilde g_{af/\delta}\tilde g_{\beta e/c} + \tilde g_{af/\delta}\tilde g_{be/\gamma}\},\\ &\tilde g^{fe}\tilde g_{af/d}\tilde g_{be/c} + \tilde g^{fe}\tilde g_{\alpha f/d}\tilde g_{\beta e/c} = \tilde g^{fe}\tilde g_{af/d}\tilde g_{be/c} - \tilde g^{\theta\varepsilon}\tilde g_{a\theta/d}\tilde g_{b\varepsilon/c} = 0,\\ &\tilde g^{fe}\tilde g_{\alpha f/\delta}\tilde g_{\beta e/\gamma} + \tilde g^{fe}\tilde g_{af/\delta}\tilde g_{be/\gamma} = -\tilde g^{\theta\varepsilon}\tilde g_{a\theta/\delta}\tilde g_{b\varepsilon/\gamma} + \tilde g^{fe}\tilde g_{af/\delta}\tilde g_{be/\gamma} = 0,\\ &\tilde g^{fe}\tilde g_{\alpha f/d}\tilde g_{be/\gamma} + \tilde g^{fe}\tilde g_{af/d}\tilde g_{\beta e/\gamma} = -\tilde g^{\theta\varepsilon}\tilde g_{a\theta/d}\tilde g_{\beta\varepsilon/\gamma} + \tilde g^{fe}\tilde g_{af/d}\tilde g_{\beta e/\gamma} = 0,\\ &\tilde g^{fe}\tilde g_{\alpha f/\delta}\tilde g_{be/c} + \tilde g^{fe}\tilde g_{af/\delta}\tilde g_{\beta e/c} = -\tilde g^{\theta\varepsilon}\tilde g_{a\theta/\delta}\tilde g_{\beta\varepsilon/c} + \tilde g^{fe}\tilde g_{af/\delta}\tilde g_{\beta e/c} = 0.\end{aligned}$$

This establishes the para-Gray identity for para-Hermitian manifolds. □

3. The Tricerri-Vanhecke curvature decomposition

3.1. **Hermitian models.** Let $(V,\langle\cdot,\cdot\rangle,J)$ be a Hermitian structure. Extend the inner product to tensors of all types. Let $\Omega(x,y) := \langle x, Jy\rangle$ be the Kaehler form and let $\mathfrak A(V)\subset\otimes^4V^*$ be the space of algebraic curvature tensors on V. Set

$$\begin{aligned}
&S^2_{0,+}(V^*,J) := \{\theta\in S^2(V^*) : J^*\theta=\theta, \theta\perp\langle\cdot,\cdot\rangle\},\\
&\Lambda^2_{0,+}(V^*,J) := \{\theta\in \Lambda^2(V^*) : J^*\theta=\theta, \theta\perp\Omega\},\\
&S^2_{-}(V^*,J) := \{\theta\in S^2(V^*) : J^*\theta=-\theta\},\\
&\Lambda^2_{-}(V^*,J) := \{\theta\in \Lambda^2(V^*) : J^*\theta=-\theta\},\\
&\mathcal{U} := \{U\in \mathrm{GL}_{\mathbb{R}}(V) : UJ=JU \quad \text{and} \quad U^*\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle\}.
\end{aligned}$$

Pullback defines a natural orthogonal action of the unitary group $\mathcal{U}$, by the orthogonal group $O(V,\langle\cdot,\cdot\rangle)$, and by the general linear group $\mathrm{GL}_{\mathbb{R}}(V)$ on $V^*\otimes V^*$ and on $\mathfrak{A}(V)$. As a $\mathrm{GL}_{\mathbb{R}}(V)$ module, there is a direct sum decomposition of

$$V^*\otimes V^* = S^2(V^*)\oplus\Lambda^2(V^*)$$

into the symmetric and the anti-symmetric 2-tensors, respectively; these modules are irreducible $\mathrm{GL}_{\mathbb{R}}(V)$ modules. $\Lambda^2(V^*)$ is an irreducible $O(V,\langle\cdot,\cdot\rangle)$ module. Let $S^2_0(V^*,\langle\cdot,\cdot\rangle)$ be the subspace of trace free symmetric 2-tensors. There is a further irreducible orthogonal decomposition of

$$S^2(V^*) = \langle\cdot,\cdot\rangle\cdot\mathbb{R}\oplus S^2_0(V^*,\langle\cdot,\cdot\rangle)\,.$$

Finally, as $\mathcal{U}$ modules, we have an orthogonal direct sum decomposition:

$$\begin{array}{rlllll}
V^*\otimes V^* & =\langle\cdot,\cdot\rangle\cdot\mathbb{R} & \oplus & S^2_{0,+}(V^*,J) & \oplus & S^2_{-}(V^*,J)\\
\oplus & \quad\Omega\cdot\mathbb{R} & \oplus & \Lambda^2_{0,+}(V^*,J) & \oplus & \Lambda^2_{-}(V^*,J)\,.
\end{array}\tag{3.a}$$

If $\theta\in V^*\otimes V^*$, let $\theta_{0,+,S}$, $\theta_{-,S}$, and $\theta_{-,\Lambda}$ denote the components of θ in $S^2_{0,+}(V^*,J)$, $S^2_{-}(V^*,J)$, and $\Lambda^2_{-}(V^*,J)$, respectively.

Let $\{e_i\}$ be a basis for V. Let $\varepsilon_{ij}:=\langle e_i,e_j\rangle$ and let ε^{ij} be the inverse matrix. Let $A\in\mathfrak{A}(V)$. Let τ, $\tau^\star$, ρ, and $\rho^\star$ be the scalar curvature, the $\star$-scalar curvature, the Ricci tensor, and the $\star$-Ricci tensor:

$$\begin{array}{ll}
\rho(x,y) := \varepsilon^{ij}A(e_i,x,y,e_j), & \tau := \varepsilon^{ij}\rho(e_i,e_j),\\
\rho^\star(x,y) := \varepsilon^{ij}A(e_i,x,Jy,Je_j), & \tau^\star := \varepsilon^{ij}\rho^\star(e_i,e_j)\,.
\end{array}\tag{3.b}$$

We refer to [8] for the proof of the following result:

Theorem 3.1. *Let $(V,\langle\cdot,\cdot\rangle,J)$ be a Hermitian structure.*

(1) *We have the following orthogonal direct sum decomposition of $\mathfrak{A}(V)$ into irreducible $\mathcal{U}$ modules:*
   (a) *If $2n=4$, $\mathfrak{A}(V)=\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4\oplus\mathcal{W}_7\oplus\mathcal{W}_8\oplus\mathcal{W}_9$.*
   (b) *If $2n=6$, $\mathfrak{A}(V)=\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4\oplus\mathcal{W}_5\oplus\mathcal{W}_7\oplus\mathcal{W}_8\oplus\mathcal{W}_9\oplus\mathcal{W}_{10}$.*
   (c) *If $2n\geq 8$, $\mathfrak{A}(V)=\mathcal{W}_1\oplus\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4\oplus\mathcal{W}_5\oplus\mathcal{W}_6\oplus\mathcal{W}_7\oplus\mathcal{W}_8\oplus\mathcal{W}_9\oplus\mathcal{W}_{10}$.*

   We have $\mathcal{W}_1\approx\mathcal{W}_4$ and, if $2n\geq 6$, $\mathcal{W}_2\approx\mathcal{W}_5$. The other $\mathcal{U}$ modules appear with multiplicity 1.
(2) *We have that:*
   (a) *$\tau\oplus\tau^\star:\mathcal{W}_1\oplus\mathcal{W}_4\xrightarrow{\approx}\mathbb{R}\oplus\mathbb{R}$.*
   (b) *If $2n=4$, $\rho_{0,+,S}:\mathcal{W}_2\xrightarrow{\approx}S^2_{0,+}(V^*,J)$.*
   (c) *If $2n\geq 6$, $\rho_{0,+,S}\oplus\rho^\star_{0,+,S}:\mathcal{W}_2\oplus\mathcal{W}_5\xrightarrow{\approx}S^2_{0,+}(V^*,J)\oplus S^2_{0,+}(V^*,J)$.*
   (d) *$\mathcal{W}_3=\{A\in\mathfrak{A}(V):A(x,y,z,w)=A(Jx,Jy,z,w)\ \forall\ x,y,z,w\}\cap\ker(\rho)$.*
   (e) *If $2n\geq 8$, $\mathcal{W}_6=\ker(\rho\oplus\rho^\star)\cap\{A\in\mathfrak{A}(V):J^*A=A\}\cap\mathcal{W}_3^\perp$.*
   (f) *$\mathcal{W}_7=\{A\in\mathfrak{A}(V):A(Jx,y,z,w)=A(x,y,Jz,w)\ \forall\ x,y,z,w\}$.*
   (g) *$\rho_{-,S}:\mathcal{W}_8\xrightarrow{\approx}S^2_{-}(V^*,J)$.*
   (h) *$\rho^\star_{-,\Lambda}:\mathcal{W}_9\xrightarrow{\approx}\Lambda^2_{-}(V^*,J)$.*
   (i) *If $2n\geq 6$, $\mathcal{W}_{10}=\{A\in\mathfrak{A}(V):J^*A=-A\}\cap\ker(\rho\oplus\rho^\star)$.*

3.2. **Para-Hermitian models.** Let $(\tilde V,\widetilde{\langle\cdot,\cdot\rangle},\tilde J)$ be a para-Hermitian structure; the metric is non-degenerate on the space of algebraic curvature tensors $\mathfrak{A}(\tilde V)$. Let $\tilde\Omega(x,y):=\widetilde{\langle x,\tilde Jy\rangle}$ be the para-Kaehler form. We have

$$\tilde J^*\tilde\Omega=-\tilde\Omega\quad\text{and}\quad\tilde J^*\widetilde{\langle\cdot,\cdot\rangle}=-\widetilde{\langle\cdot,\cdot\rangle}\,.$$

Set

$$\begin{aligned}
&S^2_+(\tilde V^*,\tilde J):=\{\theta\in S^2(\tilde V^*):\tilde J^*\theta=\theta\},\\
&\Lambda^2_+(\tilde V^*,\tilde J):=\{\theta\in \Lambda^2(\tilde V^*):\tilde J^*\theta=\theta\},\\
&S^2_{0,-}(\tilde V^*,\tilde J):=\{\theta\in S^2(\tilde V^*):\tilde J^*\theta=-\theta,\theta\perp\widetilde{\langle\cdot,\cdot\rangle}\},\\
&\Lambda^2_{0,-}(\tilde V^*,\tilde J):=\{\theta\in \Lambda^2(\tilde V^*):\tilde J^*\theta=-\theta,\theta\perp\tilde\Omega\},\\
&\tilde{\mathcal U}:=\{\tilde U\in\mathrm{GL}_{\mathbb R}(\tilde V):\tilde U\tilde J=\tilde J\tilde U\quad\text{and}\quad\tilde U^*\widetilde{\langle\cdot,\cdot\rangle}=\widetilde{\langle\cdot,\cdot\rangle}\}.
\end{aligned}$$

Fix a basis $\{\tilde e_i\}$ for $\tilde V$ and let $\tilde\varepsilon_{ij}$ be the components of the inner product relative to this basis. If $\tilde A$ is an algebraic curvature tensor, define:

$$\begin{aligned}
\rho(x,y)&:=\tilde\varepsilon^{ij}\tilde A(\tilde e_i,x,y,\tilde e_j), & \tau&:=\tilde\varepsilon^{ij}\rho(\tilde e_i,\tilde e_j),\\
\rho^\star(x,y)&:=-\tilde\varepsilon^{ij}\tilde A(\tilde e_i,x,\tilde Jy,\tilde J\tilde e_j), & \tau^\star&:=\tilde\varepsilon^{ij}\rho^\star(\tilde e_i,\tilde e_j)\,.
\end{aligned}$$

The decomposition of Equation (3.a) extends to this setting to become:

$$\begin{array}{rcccccc}
\tilde V^*\otimes\tilde V^* &=& \widetilde{\langle\cdot,\cdot\rangle}\cdot\mathbb R &\oplus& S^2_{0,-}(\tilde V^*,\tilde J) &\oplus& S^2_+(\tilde V^*,\tilde J)\\
&\oplus& \tilde\Omega\cdot\mathbb R &\oplus& \Lambda^2_{0,-}(\tilde V^*,\tilde J) &\oplus& \Lambda^2_+(\tilde V^*,\tilde J)\,.
\end{array}$$

Theorem 3.2. *Let $(\tilde V,\widetilde{\langle\cdot,\cdot\rangle},\tilde J)$ be a para-Hermitian structure.*

(1) *We have the following orthogonal direct sum decomposition of $\mathfrak{A}(\tilde V)$ into irreducible $\tilde{\mathcal U}$ modules:*
   (a) *If $2n=4$, $\mathfrak{A}(\tilde V)=\tilde{\mathcal W}_1\oplus\tilde{\mathcal W}_2\oplus\tilde{\mathcal W}_3\oplus\tilde{\mathcal W}_4\oplus\tilde{\mathcal W}_7\oplus\tilde{\mathcal W}_8\oplus\tilde{\mathcal W}_9$.*
   (b) *If $2n=6$, $\mathfrak{A}(\tilde V)=\tilde{\mathcal W}_1\oplus\tilde{\mathcal W}_2\oplus\tilde{\mathcal W}_3\oplus\tilde{\mathcal W}_4\oplus\tilde{\mathcal W}_5\oplus\tilde{\mathcal W}_7\oplus\tilde{\mathcal W}_8\oplus\tilde{\mathcal W}_9\oplus\tilde{\mathcal W}_{10}$.*
   (c) *If $2n\geq 8$, $\mathfrak{A}(\tilde V)=\tilde{\mathcal W}_1\oplus\tilde{\mathcal W}_2\oplus\tilde{\mathcal W}_3\oplus\tilde{\mathcal W}_4\oplus\tilde{\mathcal W}_5\oplus\tilde{\mathcal W}_6\oplus\tilde{\mathcal W}_7\oplus\tilde{\mathcal W}_8\oplus\tilde{\mathcal W}_9\oplus\tilde{\mathcal W}_{10}$.*

   We have $\tilde{\mathcal W}_1\approx\tilde{\mathcal W}_4$ and, if $2n\geq 6$, $\tilde{\mathcal W}_2\approx\tilde{\mathcal W}_5$. The other $\tilde{\mathcal U}$ modules appear with multiplicity 1.

(2) *We have that:*
   (a) $\tau\oplus\tau^\star:\tilde{\mathcal W}_1\oplus\tilde{\mathcal W}_4\xrightarrow{\approx}\mathbb R\oplus\mathbb R$.
   (b) *If $2n=4$, $\rho_{0,-,S}:\tilde{\mathcal W}_2\xrightarrow{\approx}S^2_{0,-}(\tilde V^*,\tilde J)$.*
   (c) *If $2n\geq 6$, $\rho_{0,-,S}\oplus\rho^\star_{0,-,S}:\tilde{\mathcal W}_2\oplus\tilde{\mathcal W}_5\xrightarrow{\approx}S^2_{0,-}(\tilde V^*,\tilde J)\oplus S^2_{0,-}(\tilde V^*,\tilde J)$.*
   (d) $\tilde{\mathcal W}_3=\{\tilde A\in\mathfrak{A}(\tilde V):\tilde A(x,y,z,w)=-\tilde A(\tilde Jx,\tilde Jy,z,w)\ \forall\ x,y,z,w\}$ $\cap\ker(\rho)$.
   (e) *If $2n\geq 8$, $\tilde{\mathcal W}_6=\ker(\rho\oplus\rho^\star)\cap\{\tilde A\in\mathfrak{A}(\tilde V):\tilde J^*\tilde A=\tilde A\}\cap\tilde{\mathcal W}_3^\perp$.*
   (f) $\tilde{\mathcal W}_7=\{\tilde A\in\mathfrak{A}(\tilde V):\tilde A(\tilde Jx,y,z,w)=\tilde A(x,y,\tilde Jz,w)\ \ \forall\ x,y,z,w\}$.
   (g) $\rho_{+,S}:\tilde{\mathcal W}_8\xrightarrow{\approx}S^2_+(\tilde V^*,\tilde J)$.
   (h) $\rho^\star_{+,\Lambda}:\tilde{\mathcal W}_9\xrightarrow{\approx}\Lambda^2_+(\tilde V^*,\tilde J)$.
   (i) *If $2n\geq 6$, $\tilde{\mathcal W}_{10}=\{\tilde A\in\mathfrak{A}(\tilde V):\tilde J^*\tilde A=-\tilde A\}\cap\ker(\rho\oplus\rho^\star)$.*

Proof. Let $(V,\langle\cdot,\cdot\rangle,J)$ be a Hermitian structure. We let $V_{\mathbb C}:=V\otimes_{\mathbb R}\mathbb C$ be the complexification of V. We extend $\langle\cdot,\cdot\rangle$ to be complex bi-linear and we extend J to be complex linear. We extend an element of $\mathfrak{A}(V)$ to be complex linear to define

$$\mathfrak{A}(V_{\mathbb C}):=\{A_{\mathbb C}\in\otimes^4V^*_{\mathbb C}:\text{Equation (1.a) holds}\}=\mathfrak{A}(V)\otimes_{\mathbb R}\mathbb C\,.$$

Let $A\in\mathfrak{A}(V)$. If $\{\xi_i\}$ is any $\mathbb C$-basis for $V_{\mathbb C}$, then Equation (3.b) remains valid where $\varepsilon_{ij}:=\langle\xi_i,\xi_j\rangle$. Let

$$\mathcal U_{\mathbb C}:=\{U\in\mathrm{GL}_{\mathbb C}(V_{\mathbb C}):JU=UJ\text{ and }U^*\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle\}\,.$$

Let $\{e_1, ..., e_n, f_1, ..., f_n\}$ be an orthonormal basis for V where

$$Je_i = f_i \quad \text{and} \quad Jf_i = -e_i\,.$$

We let

$$\begin{array}{l}\tilde{e}_i := \sqrt{-1}e_i, \quad \tilde{f}_i := -f_i, \quad \tilde{J} := \sqrt{-1}J, \quad \tilde{V} := \operatorname{Span}_{\mathbb{R}}\{\tilde{e}_i, \tilde{f}_i\},\\ \tilde{\mathcal{U}} := \{\tilde{U} \in \operatorname{GL}_{\mathbb{R}}(\tilde{V}) : \tilde{U}\tilde{J} = \tilde{J}\tilde{U} \quad \text{and} \quad \tilde{U}^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\} = \mathcal{U}_{\mathbb{C}} \cap \operatorname{GL}_{\mathbb{R}}(\tilde{V}),\\ \mathfrak{A}(\tilde{V}) = \mathfrak{A}(V_{\mathbb{C}}) \cap \otimes^4\tilde{V}^*\,.\end{array}$$

Since V has a positive definite metric, $\tilde{V}$ inherits a metric $\widetilde{\langle\cdot,\cdot\rangle}$ of neutral signature (n, n); the vectors $\tilde{e}_i$ being timelike and the vectors $\tilde{f}_i$ being spacelike. Certain sign changes now manifest themselves:

$$\rho^\star(x, y) = -\varepsilon^{ij}A(e_i, x, \tilde{J}y, \tilde{J}e_i), \quad \tau^\star = \varepsilon^{ij}\rho^\star(e_i, e_j)\,.$$

In the decomposition of Equation (3.a), we have

$$\begin{array}{rl} & S^2_\pm(V^*, J) \otimes_{\mathbb{R}} \mathbb{C} = \{\theta \in \otimes^2 V^*_{\mathbb{C}} : \theta(x, y) = \theta(y, x), \theta(Jx, Jy) = \pm\theta(x, y)\}\\ = & \{\theta \in \otimes^2\tilde{V}^*_{\mathbb{C}} : \theta(x, y) = \theta(y, x), \theta(\tilde{J}x, \tilde{J}y) = \mp\theta(x, y)\} = S^2_\mp(\tilde{V}^*, \tilde{J}) \otimes_{\mathbb{R}} \mathbb{C},\\ & \Lambda^2_\pm(V^*, J) \otimes_{\mathbb{R}} \mathbb{C} = \{\theta \in \otimes^2 V^*_{\mathbb{C}} : \theta(x, y) = -\theta(y, x), \theta(Jx, Jy) = \pm\theta(x, y)\}\\ = & \{\theta \in \otimes^2\tilde{V}^*_{\mathbb{C}} : \theta(x, y) = -\theta(y, x), \theta(\tilde{J}x, \tilde{J}y) = \mp\theta(x, y)\} = \Lambda^2_\mp(\tilde{V}^*, \tilde{J}) \otimes_{\mathbb{R}} \mathbb{C}\,.\end{array}$$

This defines a bijective correspondence which derives the decomposition of Theorem 3.2 from that of Theorem 3.1. The correspondence is reversible and hence the modules in Theorem 3.2 can not be decomposed further. □

Remark 3.3. We started in the Hermitian setting to deduce a theorem in the para-Hermitian setting. Thus the Tricerri-Vanhecke decomposition works equally well in the pseudo-Hermitian setting by changing both the inner product and the operator J. Suppose given integers p and q with $p + q = n$. By setting

$$\tilde{e}_i := \left\{\begin{array}{rl} \sqrt{-1}e_i & \text{if} \quad 1 \le i \le p\\ e_i & \text{if} \quad p < i \le n \end{array}\right\}$$
$$\tilde{f}_i := \left\{\begin{array}{rl} \sqrt{-1}f_i & \text{if} \quad 1 \le i \le p\\ f_i & \text{if} \quad p < i \le n \end{array}\right\}$$

and by taking $\tilde{J} := J$, we could create a pseudo-Hermitian model of signature $(2p, 2q)$. The analogous correspondence would then permit us to deduce a Tricerri-Vanhecke decomposition theorem in the pseudo-Hermitian signature as well.

4. Linearizing the problem

We fix a para-Hermitian structure $(\tilde{V}, \widetilde{\langle\cdot,\cdot\rangle}, \tilde{J})$ hence forth. If $\Theta \in \otimes^4\tilde{V}^*$, set

$$\mathcal{P}(\Theta)(x, y, z, w) := \Theta(x, z, y, w) + \Theta(y, w, x, z) - \Theta(x, w, y, z) - \Theta(y, z, x, w)\,.$$

Lemma 4.1. *If $\Theta \in S^2_-(\tilde{V}^*, \tilde{J}) \otimes S^2(\tilde{V}^*)$, then $\mathcal{P}(\Theta)$ is an algebraic curvature tensor such that the complex model $(\tilde{V}, \widetilde{\langle\cdot,\cdot\rangle}, \tilde{J}, \mathcal{P}(\Theta))$ is geometrically realizable by a para-Hermitian manifold.*

Proof. Let $\{e_1, ..., e_n, f_1, ..., f_n\}$ be a basis for $\mathbb{R}^{2n}$. Define an inner product Ξ of signature (n, n) on $\mathbb{R}^{2n}$ whose non-zero entries are

$$\Xi(e_1, e_1) = ... = \Xi(e_n, e_n) = -1 \quad \text{and} \quad \Xi(f_1, f_1) = ... = \Xi(f_n, f_n) = +1\,.$$

If $v \in \mathbb{R}^{2n}$, expand $v = x_1e_1 + ... + x_ne_n + y_1f_1 + ... + y_nf_n$ to define coordinates $(x_1, ..., x_n, y_1, ..., y_n) = (u_1, ..., u_{2n})$. Define

$$\tilde{\mathcal{J}}\partial_{x_1} := \partial_{y_1}, \quad ... \quad, \tilde{\mathcal{J}}\partial_{x_n} := \partial_{y_n}, \quad \tilde{\mathcal{J}}\partial_{y_1} := \partial_{x_1}, \quad ... \quad, \tilde{\mathcal{J}}\partial_{y_n} := \partial_{x_n}\,.$$

Let $\Theta \in S^2_-(\tilde V^*, \tilde J) \otimes S^2(\tilde V^*)$. Define

$$\tilde g_{ij} := \Xi_{ij} + 2\Theta_{ijkl}u^k u^l \,. \tag{4.a}$$

Since $\Theta(x,y,z,w) = -\Theta(\tilde Jx, \tilde Jy, z, w)$, $\tilde{\mathcal{J}}^*\tilde g = -\tilde g$. Let B_ϵ be the Euclidean ball of radius $\epsilon > 0$ centered at the origin. Since $\tilde g$ is non-singular at the origin, there exists $\epsilon > 0$ so $\tilde g$ is non singular on B_ϵ; let $\tilde{\mathcal{M}} := (B_\epsilon, \tilde g, \tilde{\mathcal{J}})$ be the resulting para-Hermitian manifold. Since the first derivatives of the metric vanish at 0,

$$\begin{aligned} R(\partial_{u_i}, \partial_{u_j}, \partial_{u_k}, \partial_{u_l})(0) &= \tfrac12\{\partial_{u_i}\partial_{u_k}\tilde g_{jl} + \partial_{u_j}\partial_{u_l}\tilde g_{ik} - \partial_{u_i}\partial_{u_l}\tilde g_{jk} - \partial_{u_j}\partial_{u_k}\tilde g_{il}\} \\ &= \Theta_{ikjl} + \Theta_{jlik} - \Theta_{iljk} - \Theta_{jkil} = \mathcal{P}(\Theta). \end{aligned}$$

□

5. The proof of Theorem 1.2 (2)

Let $\tilde{\mathcal{W}}_G$ be the space of algebraic curvature tensors such that the para-Gray identity holds. Let

$$\mathfrak{P} := \mathcal{P}\{S^2_-(\tilde V^*, \tilde J) \otimes S^2(\tilde V^*)\}\,.$$

$\mathfrak{P}$ and $\tilde{\mathcal{W}}_G$ are linear subspaces of $\mathfrak{A}(\tilde V)$ which are invariant under the action of the para-unitary group $\tilde{\mathcal{U}}$. The results of Section 3 reduce the proof of Theorem 1.2 (2) to showing $\mathfrak{P} = \tilde{\mathcal{W}}_G$. We begin our study with the following result:

Lemma 5.1. $\mathfrak{P} \subset \tilde{\mathcal{W}}_G \subset \tilde{\mathcal{W}}_7^\perp$.

Proof. By Lemma 4.1, every element of $\mathfrak{P}$ can be geometrically realized by a para-Hermitian manifold. Theorem 2.1 now implies $\mathfrak{P} \subset \tilde{\mathcal{W}}_G$. We show $\tilde{\mathcal{W}}_G \subset \tilde{\mathcal{W}}_7^\perp$ by showing $\tilde{\mathcal{W}}_G \cap \tilde{\mathcal{W}}_7 = \{0\}$. Let $\tilde A \in \tilde{\mathcal{W}}_G \cap \tilde{\mathcal{W}}_7$. Since $\tilde A \in \tilde{\mathcal{W}}_7$, the curvature symmetries imply additionally that

$$\begin{aligned} &\tilde A(\tilde Jx, y, z, w) = -\tilde A(\tilde Jx, y, w, z) = -\tilde A(x, y, \tilde Jw, z) = \tilde A(x, y, z, \tilde Jw) \\ &\quad = -\tilde A(y, x, z, \tilde Jw) = -\tilde A(\tilde Jy, x, z, w) = \tilde A(x, \tilde Jy, z, w). \end{aligned}$$

Since $\tilde A \in \tilde{\mathcal{W}}_G$, we have

$$\begin{aligned} 0 &= \tilde A(x,y,z,w) + \tilde A(\tilde Jx, \tilde Jy, \tilde Jz, \tilde Jw) \\ &\quad + \tilde A(\tilde Jx, \tilde Jy, z, w) + \tilde A(x, y, \tilde Jz, \tilde Jw) + \tilde A(\tilde Jx, y, \tilde Jz, w) \\ &\quad + \tilde A(x, \tilde Jy, z, \tilde Jw) + \tilde A(\tilde Jx, y, z, \tilde Jw) + \tilde A(x, \tilde Jy, \tilde Jz, w) \\ &\quad = 8\tilde A(x,y,z,w). \end{aligned}$$

□

We continue our study with:

Lemma 5.2.

1. $\tau \oplus \tau^\star : \mathfrak{P} \to \mathbb{R} \oplus \mathbb{R} \to 0$. *Thus* $\tilde{\mathcal{W}}_1 \oplus \tilde{\mathcal{W}}_4 \subset \mathfrak{P}$.
2. *If* $2n = 4$, *then* $\rho_{0,-,S} : \mathfrak{P} \to S^2_{0,-}(\tilde V^*, \tilde J) \to 0$. *Thus* $\tilde{\mathcal{W}}_2 \subset \mathfrak{P}$.
3. $\rho_{+,S} : \mathfrak{P} \to S^2_+(\tilde V^*, \tilde J) \to 0$. *Thus* $\tilde{\mathcal{W}}_8 \subset \mathfrak{P}$.
4. $\rho^\star_{+,\Lambda} : \mathfrak{P} \to \Lambda^2_+(\tilde V^*, \tilde J) \to 0$. *Thus* $\tilde{\mathcal{W}}_9 \subset \mathfrak{P}$.
5. *If* $2n \geq 6$, *then* $\{\rho_{0,-,S} \oplus \rho^\star_{0,-,S}\} : \mathfrak{P} \to \{S^2_{0,-}(\tilde V^*, \tilde J) \oplus S^2_{0,-}(\tilde V^*, \tilde J)\} \to 0$. *Thus* $\tilde{\mathcal{W}}_2 \oplus \tilde{\mathcal{W}}_5 \subset \mathfrak{P}$.
6. $\mathfrak{P} \cap \tilde{\mathcal{W}}_3 \neq \{0\}$. *Thus* $\tilde{\mathcal{W}}_3 \subset \mathfrak{P}$.
7. $\mathfrak{P} \cap \tilde{\mathcal{W}}_{10} \neq \{0\}$. *Thus* $\tilde{\mathcal{W}}_{10} \subset \mathfrak{P}$.
8. *If* $2n \geq 6$, *then* $\mathfrak{P} \cap \tilde{\mathcal{W}}_6 \neq \{0\}$. *Thus* $\tilde{\mathcal{W}}_6 \subset \mathfrak{P}$.

Proof. As in the proof of Lemma 4.1, we examine metrics $\tilde g = \Xi + O(|u|^2)$; let $\tilde A \in \mathfrak{P}$ be the curvature tensor at the origin. Set $\tilde A^*(x,y,z,w) := \tilde A(x, y, \tilde Jz, \tilde Jw)$. Let

$$\xi \circ \eta := \tfrac12(\xi \otimes \eta + \eta \otimes \xi)$$

denote the symmetric product. Let ϱ and ε be real constants. Consider the para-Hermitian metric:

$$\tilde g=\Xi-\varepsilon x_1^2(dx_1\circ dx_1-dy_1\circ dy_1)-\varrho x_1^2(dx_2\circ dx_2-dy_2\circ dy_2)\,.$$

The non-zero entries of $\tilde A$ are, up to the usual $\mathbb{Z}_2$ symmetries,

$$\begin{array}{l}\tilde A(\partial_{x_1},\partial_{y_1},\partial_{y_1},\partial_{x_1})=-\varepsilon,\\ \tilde A(\partial_{x_1},\partial_{x_2},\partial_{x_2},\partial_{x_1})=\varrho,\quad \tilde A(\partial_{x_1},\partial_{y_2},\partial_{y_2},\partial_{x_1})=-\varrho\,.\end{array}$$

Since the $\{\partial_{x_i}\}$ are timelike and the $\{\partial_{y_i}\}$ are spacelike, $\tau=2\varepsilon+4\varrho$ and $\tau^\star=2\varepsilon$ so $\tau\oplus\tau^\star$ is a surjective map from $\mathfrak{P}$ to $\mathbb{R}\oplus\mathbb{R}$. Thus Assertion (1) follows from Theorem 3.2:

$$\tilde{\mathcal{W}}_1\oplus\tilde{\mathcal{W}}_4\subset\mathfrak{P}\,.$$

The non-zero entries in the Ricci tensor are given by:

$$\begin{array}{ll}\rho(\partial_{x_1},\partial_{x_1})=-\varepsilon-2\varrho, & \rho(\partial_{y_1},\partial_{y_1})=\varepsilon,\\ \rho(\partial_{x_2},\partial_{x_2})=-\varrho, & \rho(\partial_{y_2},\partial_{y_2})=\varrho\,.\end{array}$$

We take $\varrho=-1$ and $\varepsilon=2$ to ensure ρ is trace free and symmetric. We then have

$$\begin{array}{ll}\rho_{+,S}(\partial_{x_1},\partial_{x_1})=1, & \rho_{0,-,S}(\partial_{x_1},\partial_{x_1})=-1,\\ \rho_{+,S}(\partial_{y_1},\partial_{y_1})=1, & \rho_{0,-,S}(\partial_{y_1},\partial_{y_1})=1,\\ \rho_{+,S}(\partial_{x_2},\partial_{x_2})=0, & \rho_{0,-,S}(\partial_{x_2},\partial_{x_2})=1,\\ \rho_{+,S}(\partial_{y_2},\partial_{y_2})=0, & \rho_{0,-,S}(\partial_{y_2},\partial_{y_2})=-1\,.\end{array}$$

This shows that $\rho_{0,-,S}$ is non-zero on $\mathfrak{P}$; Assertion (2) now follows if $2n=4$ since $\tilde{\mathcal{W}}_5$ is not present:

$$\tilde{\mathcal{W}}_2\subset\mathfrak{P}\,.$$

It also shows $\rho_{+,S}$ is non-trivial on $\mathfrak{P}$ and establishes Assertion (3):

$$\tilde{\mathcal{W}}_8\subset\mathfrak{P}\,.$$

We clear the previous notation and consider:

$$\tilde g=\Xi-4\varepsilon x_1^2(-dx_1\circ dx_2+dy_1\circ dy_2)\,.$$

There is only one non-zero curvature entry $\tilde A(\partial_{x_1},\partial_{y_1},\partial_{y_2},\partial_{x_1})=2\varepsilon$. We have:

$$\begin{array}{ll}\tilde A^*(\partial_{x_1},\partial_{y_1},\partial_{x_2},\partial_{y_1})=2\varepsilon, & \tilde A^*(\partial_{y_2},\partial_{x_1},\partial_{y_1},\partial_{x_1})=2\varepsilon,\\ \rho^\star(\partial_{x_1},\partial_{x_2})=2\varepsilon, & \rho^\star(\partial_{y_2},\partial_{y_1})=-2\varepsilon,\\ \rho^\star_\Lambda(\partial_{x_1},\partial_{x_2})=-\rho^\star_\Lambda(\partial_{x_2},\partial_{x_1})=\varepsilon, & \rho^\star_\Lambda(\partial_{y_2},\partial_{y_1})=-\rho^\star_\Lambda(\partial_{y_1},\partial_{y_2})=-\varepsilon,\\ \rho^\star_S(\partial_{x_1},\partial_{x_2})=\rho^\star_S(\partial_{x_2},\partial_{x_1})=\varepsilon, & \rho^\star_S(\partial_{y_1},\partial_{y_2})=\rho^\star_S(\partial_{y_2},\partial_{y_1})=-\varepsilon\,.\end{array}$$

This shows $\rho^\star_{+,\Lambda}=\rho^\star_\Lambda\neq0$ so $\tilde A$ has a non-trivial component in $\tilde{\mathcal{W}}_9$. This completes the proof of Assertion (4):

$$\tilde{\mathcal{W}}_9\subset\mathfrak{P}\,.$$

Assume $2n\geq6$. We clear the previous notation and consider:

$$\tilde g=\Xi-2\varrho x_1^2(-dx_1\circ dx_2+dy_1\circ dy_2)-2\varepsilon x_1^2(-dx_2\circ dx_3+dy_2\circ dy_3)\,.$$

The non-zero curvatures now become:

$$\begin{array}{l}\tilde A(\partial_{x_1},\partial_{y_1},\partial_{y_2},\partial_{x_1})=\varrho,\\ \tilde A(\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{x_1})=-\varepsilon,\quad \tilde A(\partial_{x_1},\partial_{y_2},\partial_{y_3},\partial_{x_1})=\varepsilon\,.\end{array}$$

Note that ρ is always symmetric. We have

$$\begin{array}{ll}\rho(\partial_{y_1},\partial_{y_2})=-\varrho, & \rho(\partial_{x_1},\partial_{x_2})=0,\\ \rho(\partial_{x_2},\partial_{x_3})=\varepsilon, & \rho(\partial_{y_2},\partial_{y_3})=-\varepsilon\,.\end{array}$$

This leads to the decomposition:

$$\begin{aligned}
\rho_{0,-,S}(\partial_{x_1},\partial_{x_2}) &= \tfrac{1}{2}\varrho, \quad \rho_{+,S}(\partial_{x_1},\partial_{x_2}) = -\tfrac{1}{2}\varrho,\\
\rho_{0,-,S}(\partial_{y_1},\partial_{y_2}) &= -\tfrac{1}{2}\varrho, \quad \rho_{+,S}(\partial_{y_1},\partial_{y_2}) = -\tfrac{1}{2}\varrho,\\
\rho_{0,-,S}(\partial_{x_2},\partial_{x_3}) &= \varepsilon, \quad \rho_{+,S}(\partial_{x_2},\partial_{x_3}) = 0,\\
\rho_{0,-,S}(\partial_{y_2},\partial_{y_3}) &= -\varepsilon, \quad \rho_{+,S}(\partial_{y_2},\partial_{y_3}) = 0.
\end{aligned}$$

We have:

$$\begin{aligned}
&\tilde{A}^*(\partial_{x_1},\partial_{y_1},\partial_{x_2},\partial_{y_1}) = \varrho, &&\tilde{A}^*(\partial_{y_2},\partial_{x_1},\partial_{y_1},\partial_{x_1}) = \varrho,\\
&\tilde{A}^*(\partial_{x_1},\partial_{x_2},\partial_{y_3},\partial_{y_1}) = -\varepsilon, &&\tilde{A}^*(\partial_{x_3},\partial_{x_1},\partial_{y_1},\partial_{y_2}) = -\varepsilon,\\
&\tilde{A}^*(\partial_{x_1},\partial_{y_2},\partial_{x_3},\partial_{y_1}) = \varepsilon, &&\tilde{A}^*(\partial_{y_3},\partial_{x_1},\partial_{y_1},\partial_{x_2}) = \varepsilon\,.
\end{aligned}$$

Consequently $\rho^\star(\partial_{x_1},\partial_{x_2}) = \varrho$ and $\rho^\star(\partial_{y_2},\partial_{y_1}) = -\varrho$. This yields:

$$\begin{aligned}
\rho^\star_{0,-,S}(\partial_{x_1},\partial_{x_2}) &= \tfrac{1}{2}\varrho, \quad \rho^\star_{+,\Lambda}(\partial_{x_1},\partial_{x_2}) = \tfrac{1}{2}\varrho,\\
\rho^\star_{0,-,S}(\partial_{y_1},\partial_{y_2}) &= -\tfrac{1}{2}\varrho, \quad \rho^\star_{+,\Lambda}(\partial_{y_1},\partial_{y_2}) = +\tfrac{1}{2}\varrho.
\end{aligned}$$

If we take $\varrho = 0$ and $\varepsilon \neq 0$, then $\rho_{0,-,S} \neq 0$ and $\rho^\star_{0,-,S} = 0$. Thus

$$\begin{aligned}
&\{S^2_{0,-}(\tilde{V}^*,\tilde{J}) \oplus 0\} \cap \{\rho_{0,-,S} \oplus \rho^\star_{0,-,S}\}\mathfrak{P} \neq \{0\} \quad \text{so}\\
&\{S^2_{0,-}(\tilde{V}^*,\tilde{J}) \oplus 0\} \subset \{\rho_{0,-,S} \oplus \rho^\star_{0,-,S}\}\mathfrak{P}\,.
\end{aligned}$$

On the other hand, if we take $\varrho \neq 0$, then $\rho^\star_{0,-,S} \neq 0$. Thus we have a non-zero component in the second factor and

$$\{S^2_{0,-}(\tilde{V}^*,\tilde{J}) \oplus S^2_{0,-}(\tilde{V}^*,\tilde{J})\} \subset \{\rho_{0,-,S} \oplus \rho^\star_{0,-,S}\}\mathfrak{P}\,.$$

This establishes Assertion (5):

$$\tilde{\mathcal{W}}_2 \oplus \tilde{\mathcal{W}}_5 \subset \mathfrak{P}\,.$$

To prove Assertion (6), we consider the metric

$$\tilde{g} = \Xi - 2\{x_1^2 - y_1^2 - x_2^2 + y_2^2\}(-dx_1 \circ dx_2 + dy_1 \circ dy_2)\,.$$

The non-zero components of $\tilde{A}$ are then given, up to the usual $\mathbb{Z}_2$ symmetries by:

$$\begin{aligned}
&\tilde{A}(\partial_{x_1},\partial_{y_1},\partial_{y_2},\partial_{x_1}) = 1, &&\tilde{A}(\partial_{y_1},\partial_{x_1},\partial_{x_2},\partial_{y_1}) = 1,\\
&\tilde{A}(\partial_{x_2},\partial_{y_1},\partial_{y_2},\partial_{x_2}) = -1, &&\tilde{A}(\partial_{y_2},\partial_{x_1},\partial_{x_2},\partial_{y_2}) = -1\,.
\end{aligned}$$

We have $\rho = 0$ and $\tilde{A}(\tilde{J}x,\tilde{J}y,z,w) = -\tilde{A}(x,y,z,w)$ for all x, y, z, and w. This shows $\tilde{A} \in \tilde{\mathcal{W}}_3$ and proves Assertion (6) by showing

$$\tilde{\mathcal{W}}_3 \subset \mathfrak{P}\,.$$

Let $2n \geq 6$. We consider

$$\tilde{g} = \Xi - 2\{x_1^2 + y_1^2\}(-dx_2 \circ dx_3 + dy_2 \circ dy_3)\,.$$

The non-zero curvatures are then

$$\begin{aligned}
&\tilde{A}(\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{x_1}) = -1, &&\tilde{A}(\partial_{x_1},\partial_{y_2},\partial_{y_3},\partial_{x_1}) = 1,\\
&\tilde{A}(\partial_{y_1},\partial_{x_2},\partial_{x_3},\partial_{y_1}) = -1, &&\tilde{A}(\partial_{y_1},\partial_{y_2},\partial_{y_3},\partial_{y_1}) = 1\,.
\end{aligned}$$

We have $\rho = \rho^\star = 0$. Since $\tilde{J}^*\tilde{A} = -\tilde{A}$, $\tilde{A} \in \tilde{\mathcal{W}}_{10}$; Assertion (7) follows since

$$\tilde{\mathcal{W}}_{10} \subset \mathfrak{P}\,.$$

Let $2n \geq 8$. We take

$$\tilde{g} = \Xi - 4\{x_1x_2 + y_1y_2\}(-dx_3 \circ dx_4 + dy_3 \circ dy_4)\,.$$

The non-zero curvatures are

$$\begin{aligned}
&\tilde{A}(\partial_{x_1},\partial_{x_3},\partial_{x_4},\partial_{x_2}) = \tilde{A}(\partial_{y_1},\partial_{x_3},\partial_{x_4},\partial_{y_2})\\
=\ &\tilde{A}(\partial_{x_1},\partial_{x_4},\partial_{x_3},\partial_{x_2}) = \tilde{A}(\partial_{y_1},\partial_{x_4},\partial_{x_3},\partial_{y_2}) = -1,\\
&\tilde{A}(\partial_{x_1},\partial_{y_3},\partial_{y_4},\partial_{x_2}) = \tilde{A}(\partial_{y_1},\partial_{y_3},\partial_{y_4},\partial_{y_2})\\
=\ &\tilde{A}(\partial_{x_1},\partial_{y_4},\partial_{y_3},\partial_{x_2}) = \tilde{A}(\partial_{y_1},\partial_{y_4},\partial_{y_3},\partial_{y_2}) = 1\,.
\end{aligned}$$

We observe that $\rho = \rho^\star = 0$. Since $\tilde{A}(\tilde{J}x, \tilde{J}y, z, w) \neq -\tilde{A}(x, y, z, w)$, $\tilde{A} \notin \tilde{\mathcal{W}}_3$. Thus $\tilde{A}$ has a non-zero component in $\tilde{\mathcal{W}}_6 \oplus \tilde{\mathcal{W}}_7$. As $\mathfrak{P} \perp \tilde{\mathcal{W}}_7$, $\tilde{A}$ has a non-zero component in $\tilde{\mathcal{W}}_6$ and Assertion (8) follows; $\tilde{\mathcal{W}}_6 \subset \mathfrak{P}$. $\square$

Proof of Theorem 1.2 (2). By Lemma 5.1, we have $\mathfrak{P} \subset \tilde{\mathcal{W}}_G \subset \tilde{\mathcal{W}}_7^\perp$. The assertion $\tilde{\mathcal{W}}_7^\perp \subset \mathfrak{P}$ follows from the Tricerri-Vanhecke decomposition described in Theorem 3.2 and from Lemma 5.2. $\square$

Proof of Remark 1.3. The construction given above yields $\tilde{\mathcal{M}}$ with $d\tilde{\Omega}_P = 0$ realizing the given complex curvature model $\tilde{\mathfrak{C}}$ at P. Imposing the para-Kaehler identity $d\tilde{\Omega} \equiv 0$ globally would imply that $\tilde{R} \in \tilde{\mathcal{W}}_1 \oplus \tilde{\mathcal{W}}_2 \oplus \tilde{\mathcal{W}}_3$ so this is not possible in general. In [1], we considered a further variation

$$\tilde{h} := \Xi + 2\xi(dx_1 \circ dx_1 - dy_1 \circ dy_1) + 2\eta(dx_2 \circ dx_2 - dy_2 \circ dy_2)$$

where $\{\xi, \eta\}$ are smooth functions vanishing to second order at P. We showed it was possible to choose $\{\xi, \eta\}$ so that the resulting metric had constant scalar curvature and constant $\star$-scalar curvature. Since $\{\xi, \eta\}$ vanish to second order, $(\tilde{M}, \tilde{h}, \tilde{\mathcal{J}})$ realizes $\tilde{\mathfrak{C}}$ at P as well and $d\tilde{\Omega}_{\xi,\eta} = 0$. This establishes Remark 1.3. $\square$

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